Heavy Tails: Performance Models and Scheduling Disciplines

Part III – Delay Asymptotics

Sindo Núñez-Queija

Heavy Tails: Performance Models and Scheduling Disciplines Part III – Delay Asymptotics

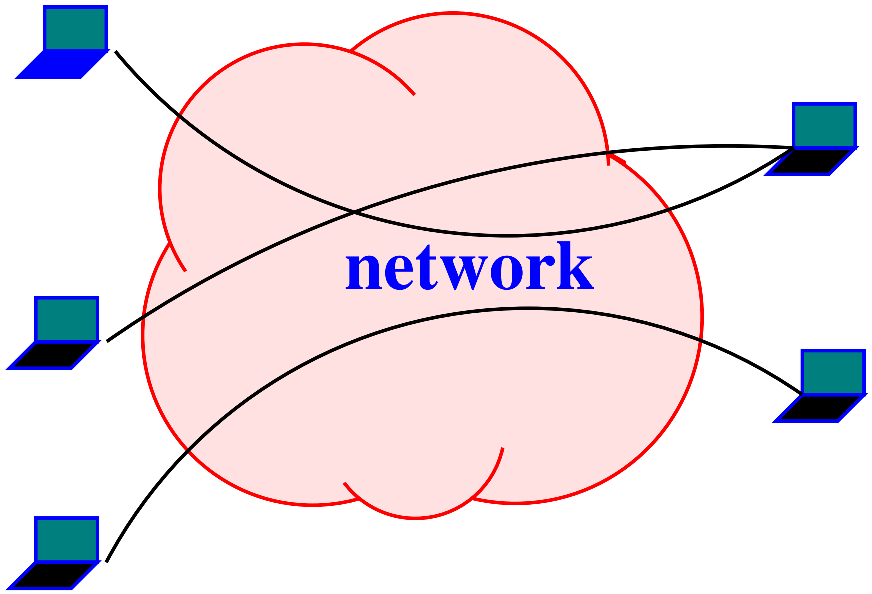
- Processor Sharing model for TCP-like traffic

- Tail asymptotics via conditional moments

- Other disciplines: FBPS, SRPT and LCFS-NP

- Differentiated services: GPS with two classes (Sem Borst, Miranda van Uitert, RNQ)

Distributed Congestion Control in Data Networks: TCP



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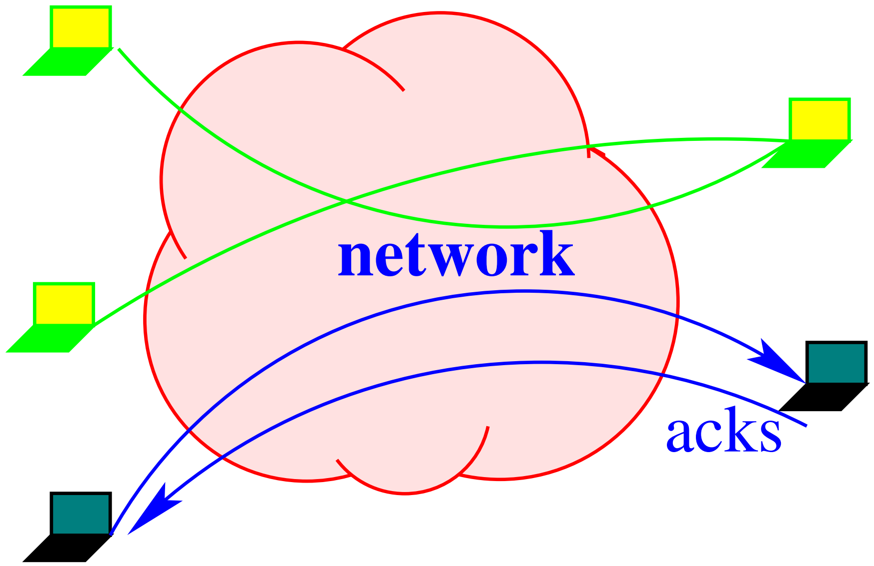

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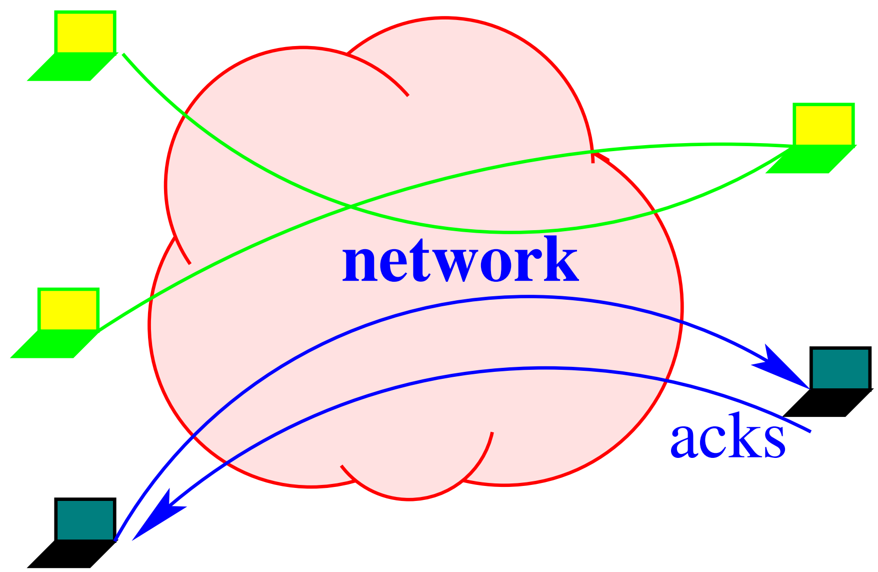
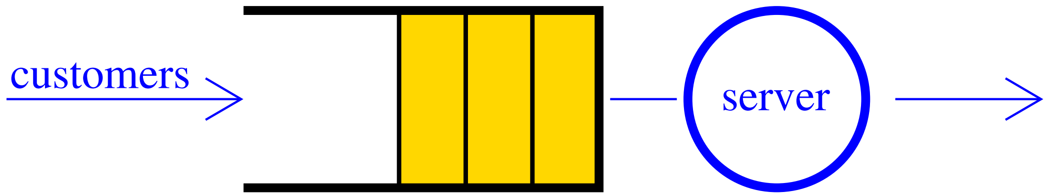


- separation of time scales
- simultaneous resource sharing

M/G/1 Processor Sharing

- flows are "initiated" according to a Poisson process of rate λ
- arbitrary flow size distribution



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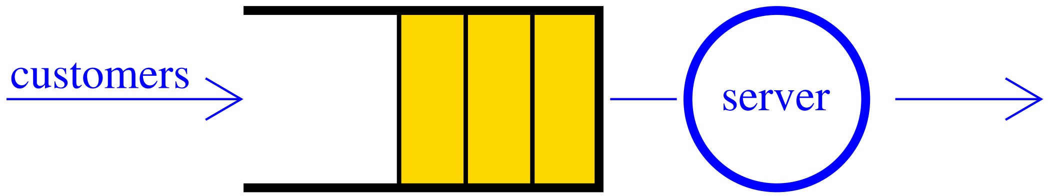

- throughput of each flow: $\frac{1}{N(t)}$
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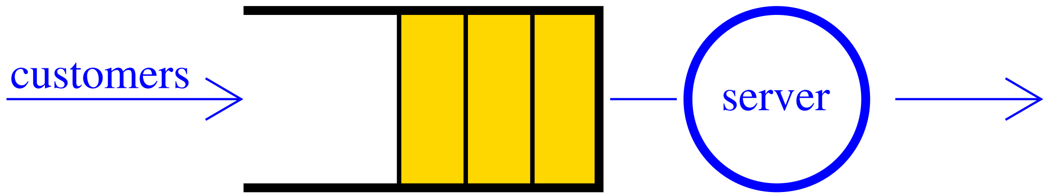

- throughput of each flow: $\frac{1}{N(t)}$
- "workload" in the queue does not correspond to workload in a buffer
- $\mathbf{P}\{N = n\} = (1-\rho)\rho^n$ with $\rho = \lambda \mathbf{E}[F]$
- $\mathbf{E}[S \,|\, F = \tau] = \frac{\tau}{1-\rho}$ $\Rightarrow \mathbf{E}[S] = \frac{\mathbf{E}[F]}{1-\rho}$
- $P(S > x)$ delay asymptotics, $x \to \infty$

Intuition

$$\lim_{x\to\infty} \frac{\mathbf{P}\left\{S > \frac{x}{c-\rho}\right\}}{\mathbf{P}\left\{B > x\right\}} = 1$$

$c =$ **average service capacity**

$\rho =$ **traffic load (**$\rho < c$**)**

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"For large x, the probability that a customer's sojourn time exceeds the value $\frac{x}{c-\rho}$ is asymptotically equal to the probability that a customer's service requirement exceeds the value x"

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"For large x, the probability that a customer's sojourn time exceeds the value $\frac{x}{c-\rho}$ is asymptotically equal to the probability that a customer's service requirement exceeds the value x"

We show that this is true because (asymptotically) the two events can only occur simultaneously

intuition . . .

$S(\tau) =$ sojourn time of a customer with service requirement τ (in steady state)

When $S(\tau)$ is "large"

$$\frac{\tau}{S(\tau)} \approx c - \rho$$

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We show something stronger: a one-to-one correspondence between “large” sojourn times and “large” service requirements

$$\frac{S(\tau)}{\tau} \xrightarrow{\mathsf{P}} \frac{1}{c - \rho}$$

“fast enough” as $\tau \to \infty$.

Proof of tail equivalence

Service requirement distribution $B(x) := \mathrm{P}\{B \leq x\}$, $x \geq 0$
Tail distribution $\overline{B}(x) := 1 - B(x) = \mathrm{P}\{B > x\}$

Assumption (Service requirements)
$\overline{B}(x) \in \mathcal{IR}$

$$\liminf_{\varepsilon \downarrow 0} \liminf_{x \to \infty} \frac{\overline{B}(x(1+\varepsilon))}{\overline{B}(x)} = 1$$

$\Longrightarrow \overline{B}(x)$ **"dominates" a Pareto tail:**

$$\frac{\overline{B}(x_2)}{\overline{B}(x_1)} \geq \eta \left(\frac{x_2}{x_1}\right)^{-\zeta}$$

proof of tail equivalence . . .

If $\overline{B}(x) \in \mathcal{IR}$ then the sojourn times have the following properties

- $S(\tau)$: **stochastically increasing in** $\tau \geq 0$

- $\mathbf{E}[S(\tau)] \ = \ \frac{\tau}{c-\rho} + o(\tau)$

- $\mathbf{E}\left[\left|S(\tau) - \mathbf{E}[S(\tau)]\right|^{\kappa}\right] \ = \ o(\tau^{\kappa-\delta}), \qquad \kappa > \zeta, \delta > 0$

proof of tail equivalence . . .

- $S(\tau) \uparrow$ **in** τ**: sample-path argument**

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- $\mathbf{E}\left[S(\tau)\right] = \dfrac{\tau}{1-\rho}$

- **Variance: distinguish two cases**

  ◇ $\mathbf{E}\left[B^2\right] < \infty$

  ◇ $\mathbf{E}\left[B^\alpha\right] < \infty$ **and** $\mathbf{E}\left[B^\zeta\right] = \infty$, $1 < \alpha < \zeta < 2$

proof of tail equivalence ...

- $\mathbf{E}\left[B^2\right] < \infty$: **For** $k = 2, 3, \ldots$,

$$\mathbf{E}\left[S(\tau)^k\right] \;=\; \left(\frac{\tau}{c-\rho}\right)^k + O(\tau^{k-1}), \qquad \tau \to \infty$$

Take $\kappa > \zeta$ **and** $\delta \in (0, 1)$

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- $\mathbf{E}\left[B^\alpha\right] < \infty$ and $\mathbf{E}\left[B^\zeta\right] = \infty$, $1 < \alpha < \zeta < 2$

$$\mathbf{Var}\left[S(\tau)\right] \sim \int_{u=0}^{\tau} (\tau - u)\mathbf{P}\left\{W_{\lambda,B} > u\right\} \mathrm{d}u$$

Take $\kappa = 2$ and $\delta < \alpha - 1$

“Tail equivalence”

Theorem

If $\overline{B}(x) \in \mathcal{IR}$ *and* $S(\tau)$ *satisfies the three properties then*

$$\lim_{x \to \infty} \frac{\mathbf{P}\left\{S > \frac{x}{c-\rho}\right\}}{\mathbf{P}\left\{B > x\right\}} = 1$$

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- **When B is “small”, S can not be “large”**

- **When B is “large”, S must be “large” as well**

proof of tail equivalence . . .

When B is "small", S can not be "large":

$$\lim_{x\to\infty} \frac{\mathbf{P}\left\{S > \frac{x}{c-\rho}; B \leq x(1-\varepsilon)\right\}}{\mathbf{P}\left\{B > x(1-\varepsilon)\right\}} = 0$$

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Proof Markov's inequality:

$$\mathbf{P}\{S(\tau) - \mathbf{E}[S(\tau)] > t\} \leq \frac{\mathbf{E}\left[\left|S(\tau) - \mathbf{E}[S(\tau)]\right|^{\kappa}\right]}{t^{\kappa}}, \qquad \tau \geq 0,\ t > 0$$

$$
\begin{aligned}
\mathbf{P}\left\{S > \frac{x}{c-\rho}; B \leq x(1-\varepsilon)\right\} &= \int_{\tau=0}^{x(1-\varepsilon)} \mathbf{P}\left\{S(\tau) > \frac{x}{c-\rho}\right\} \mathrm{d}B(\tau) \\
&\leq \int_{\tau=0}^{x(1-\varepsilon)} \frac{\mathbf{E}\left[\left|S(\tau) - \mathbf{E}\left[S(\tau)\right]\right|^{\kappa}\right]}{\left(\frac{x}{c-\rho} - \mathbf{E}\left[S(x(1-\varepsilon))\right]\right)^{\kappa}} \mathrm{d}B(\tau) \\
&\leq (1+o(1)) \frac{\overline{B}(x(1-\varepsilon))\,(x(1-\varepsilon))^{\kappa-\delta}}{\left(\frac{x\varepsilon}{c-\rho}\right)^{\kappa}}
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Dividing by $\overline{B}(x(1-\varepsilon))$, and letting $x \to \infty$, proves

$$
\lim_{x\to\infty} \frac{\mathbf{P}\left\{S > \frac{x}{c-\rho}; B \le x(1-\varepsilon)\right\}}{\mathbf{P}\left\{B > x(1-\varepsilon)\right\}} = 0
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$\square$

proof of tail equivalence . . .

When B is "large", S is "large":

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Proof It suffices to show that the lim inf is ≥ 1.

$$\begin{aligned}
&\mathbf{P}\left\{S > \frac{x}{c-\rho}; B > x(1+\varepsilon)\right\} \\
&\quad = \int_{\tau=x(1+\varepsilon)}^{\infty} \mathbf{P}\left\{S(\tau) > \frac{x}{c-\rho}\right\} \mathrm{d}B(\tau) \\
&\quad \geq \underbrace{\mathbf{P}\left\{S(x(1+\varepsilon)) > \frac{x}{c-\rho}\right\}}_{\longrightarrow 1} \mathbf{P}\left\{B > x(1+\varepsilon)\right\}.
\end{aligned}$$

$\square$

Proof of tail equivalence . . .

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$$\lim_{x\to\infty} \frac{\mathbf{P}\left\{S > \frac{x}{c-\rho}\right\}}{\mathbf{P}\left\{B > x\right\}} = 1$$

Proof Part (i). 'Small B'

$$\mathbf{P}\left\{S > \frac{x}{c-\rho}\right\} \leq \underbrace{\mathbf{P}\left\{S > \frac{x}{c-\rho}; B \leq x(1-\varepsilon)\right\}}_{\text{negligible}} + \mathbf{P}\left\{B > x(1-\varepsilon)\right\}$$

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In both parts use $\overline{B}(x) \in \mathcal{IR}$ □

FBPS and SRPT

Lemma *If* $\overline{B}(x) \in \mathcal{IR}$, $\mathbf{E}[B^{\alpha}] < \infty$ *and* $\mathbf{E}\left[B^{\zeta}\right] = \infty$, *for some* $1 < \alpha < \zeta < 2$, *then* $S(\tau)$ *satisfies the three properties with* $\kappa = 2$ *and* $\delta < \alpha - 1$.

Proof [FBPS] Yashkov (1987)

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- $\mathbf{Var}\left[S(\tau)\right] = \dfrac{\lambda h_3(\tau)}{3(1-\lambda h_1(\tau))^3} + \dfrac{\lambda \tau h_2(\tau)}{(1-\lambda h_1(\tau))^3} + \dfrac{3(\lambda h_2(\tau))^2}{4(1-\lambda h_1(\tau))^4}$

- **with** $h_j(\tau) = j\int_{x=0}^{\tau} x^{j-1}\overline{B}(x)\mathrm{d}x$

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- **with** $h_j(\tau) = j \int_{x=0}^{\tau} x^{j-1}\overline{B}(x)\mathrm{d}x$

- $h_1(\tau) \to \beta_1$ **and** $h_j(\tau) = \mathrm{o}(\tau^{j-\alpha+\varepsilon})$, $j = 2, 3$ $\Box$

Proof [SRPT] Schrage and Miller (1966)

The waiting time is negligible compared to the residence time $R(\tau)$

$$\mathbf{E}\left[R(\tau)\right] = \int_{t=0}^{\tau} \frac{1}{1-\rho(t)} \mathrm{d}t,$$

$$\mathbf{Var}\left[R(\tau)\right] = \lambda \int_{t=0}^{\tau} \frac{\int_{u=0}^{t} u^2 \mathrm{d}B(u)}{(1-\rho(t))^3} \mathrm{d}t,$$

with

$$\rho(\tau) := \lambda \int_{t=0}^{\tau} t \mathrm{d}B(t).$$

□

Last-Come First-Served Non-Preemptive

Focus on

$S_{LNP}(\tau) =$ **sojourn time of a customer that arrives when the remaining service requirement of the customer in service equals** τ

$$S_{LNP}(\tau) = \tau + \sum_{n=1}^{N(\tau)} P_n + B$$

$N(\tau) =$ **number of arrived customers during the remaining service requirement** τ

$P_n =$ **i.i.d. sequence of** $M/G/1$ **busy periods**

$B =$ **customer's own service requirement**

last-come first-served non-preemptive (2)

$$\mathbf{P}\{S_{LNP} > t\} = (1-\rho)\,\mathbf{P}\{B > t\} + \rho\,\mathbf{P}\{S_{LNP}(B^r) > t\}, \qquad t \geq 0$$

B^r = **unconditional residual service requirement of the customer in service**

$$\mathbf{P}\{S_{LNP}(B^r) > t\} = \int_{\tau=0}^{\infty} \mathbf{P}\{S_{LNP}(\tau) > t\} \mathrm{d}B^r(\tau), \qquad t \geq 0$$

Show for non-integer $\nu > 2$, **if** $B(\cdot) \in \mathcal{R}_{-\nu}$ **and, hence,** $B^r(\cdot) \in \mathcal{R}_{-\alpha}$ **with** $\alpha := \nu - 1$, **then**

$$\mathbf{P}\{S_{LNP}(B^r) > \frac{x}{1-\rho}\} \sim \mathbf{P}\{B^r > x\}.$$

and

$$\mathbf{P}\{S_{LNP} > \frac{x}{1-\rho}\} \sim \rho\,\mathbf{P}\{B^r > x\},$$

Differential Equations

Let $m < \nu < m+1$, **for some integer** $m \geq 2$

$\mathbf{E}\{B^m\} < \infty$ **and** $\mathbf{E}\{P^m\} < \infty$

As $\Delta \to 0$

$$\begin{aligned}\mathbf{E}\{\left[S_{LNP}(\tau+\Delta)^k\right]\} &= (1-\lambda\Delta)\mathbf{E}\{\left[(\Delta + S_{LNP}(\tau))^k\right]\} \\ &\quad +\lambda\Delta\mathbf{E}\{\left[(\Delta + S_{LNP}(\tau) + P)^k\right]\} + o(\Delta)\end{aligned}$$

$$\frac{d}{d\tau}\mathbf{E}\{S_{LNP}(\tau)^k\} = \frac{k}{1-\rho}\mathbf{E}\{S_{LNP}(\tau)^{k-1}\} + \lambda\sum_{j=0}^{k-2}\binom{k}{j}\mathbf{E}\{S_{LNP}(\tau)^j\}\mathbf{E}\{P^{k}$$

with $\mathbf{E}\{S(0)^k\} = \mathbf{E}\{B^k\}$

differential equations (2)

$$\mathbf{E}\{S_{LNP}(\tau)\} = \beta + \frac{\tau}{1-\rho}$$

$$\mathbf{E}\{S_{LNP}(\tau)^k\} = \left(\frac{\tau}{1-\rho}\right)^k + p_{k-1}(\tau)$$

$p_{k-1}(\tau) =$ **polynomial in** τ **of degree** $k-1$

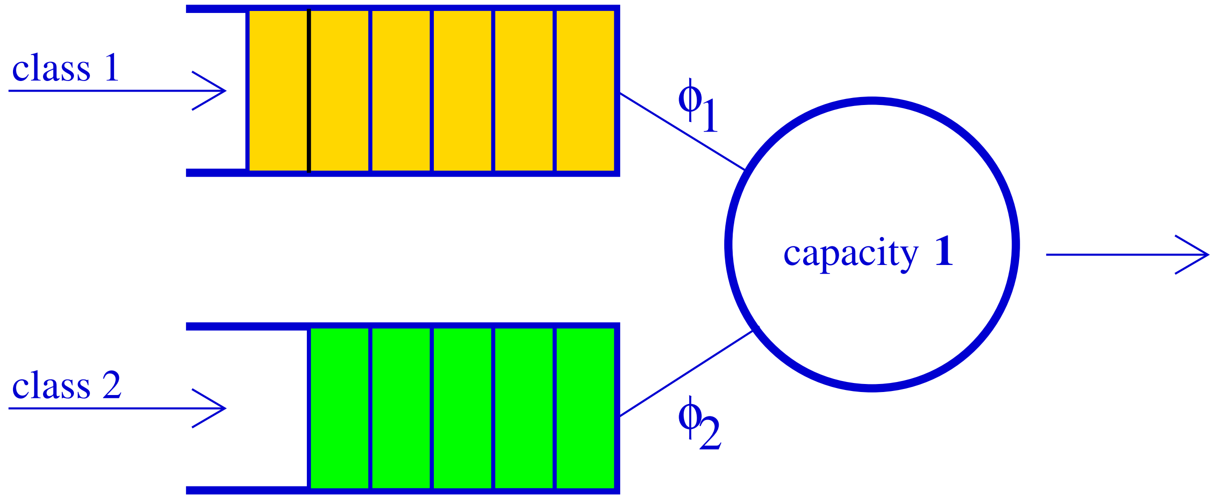
Take $\kappa = m$ **and use**

$$\mathbf{P}\{S(\tau) > t\} \leq \frac{\mathbf{E}\{S(\tau)^\kappa\} - (\mathbf{E}\{S(\tau)\})^\kappa}{(t - \mathbf{E}\{S(\tau)\})^\kappa}, \qquad \tau \geq 0,\ t > \mathbf{E}\{S(\tau)\}$$

where $\kappa \geq 2$ $\square$

Differentiated Services

Two classes



- **Between classes: GPS (Generalized Processor Sharing)**
  - ◇ $\phi_1 + \phi_2 = 1$

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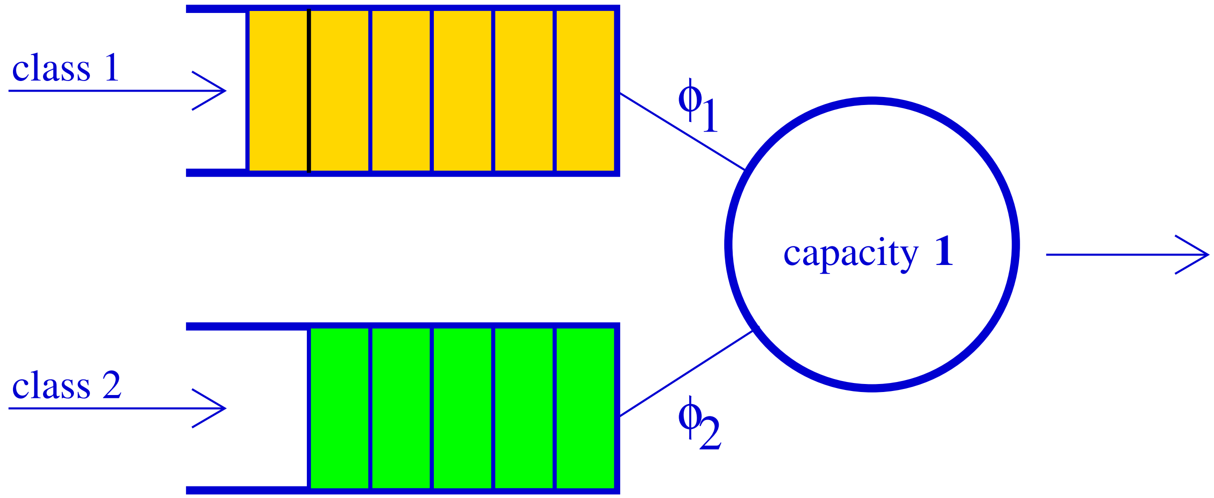

- **Between classes: GPS (Generalized Processor Sharing)**
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- **Within class 1: PS (Processor Sharing)**
  - **rate per flow between** ϕ_1/n_1 **and** $1/n_1$

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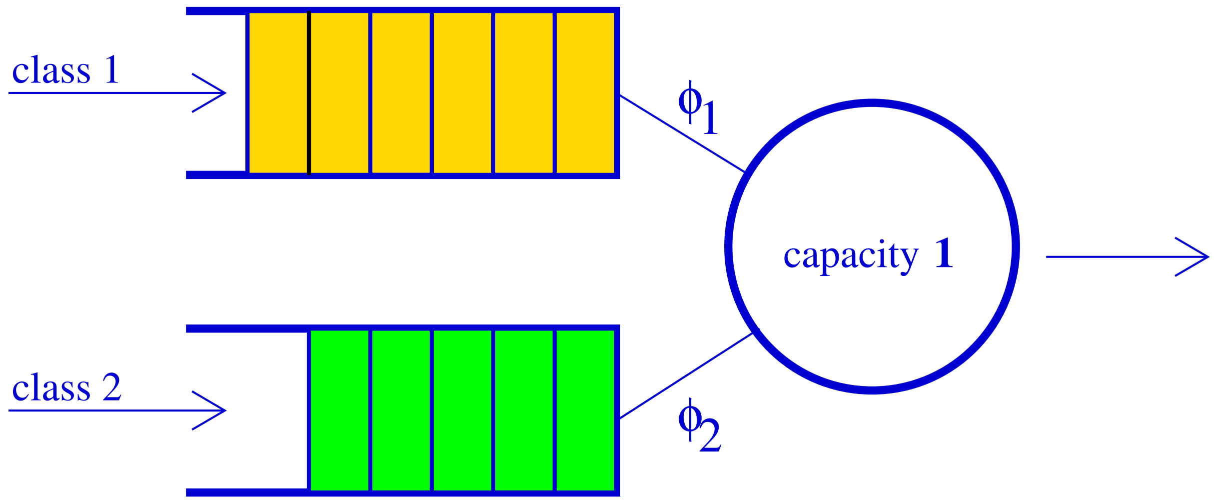

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- **Within class 1: PS (Processor Sharing)**
  - **rate per flow between** ϕ_1/n_1 **and** $1/n_1$
- **Within class 2: work-conserving, for instance PS**

Strict priorities ($\phi_1 = 0$)

M/G/1 PS with random service interruptions

- Poisson arrivals with rate λ

- Service requirement distribution $B(x) \in \mathcal{IR}$

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  - ◇ "on-periods": exponential with mean $1/\nu$
  - ◇ "off-periods": general with finite moments m_1, m_2, m_3
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  - ◇ mean service rate: $c = \frac{1}{1+\nu m_1}$
- Using method based on conditional moments
- $\mathrm{P}\{V > x\} \sim \mathrm{P}\{B > x(c-\rho)\}$

Uniform stability ($\rho_1 < \phi_1$)

- **Class-1 traffic:**
  - ◇ **Poisson arrival process of rate** λ_1
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- $A_i(s,t)$ **= class-i traffic generated during** $(s,t]$

$$\lim_{t\to\infty} \frac{1}{t-s} A_i(s,t) = \lim_{u\to-\infty} \frac{1}{s-u} A_i(u,s) = \rho_i, \ \forall s, \ \textbf{w.p. 1}$$

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Theorem:

$$\mathbf{P}\{\mathbf{S}_0 > t\} \sim \mathbf{P}\{\mathbf{B}_0 > (1-\psi_2-\rho_1)t\} \sim \mathbf{P}\{\mathbf{S}_0^{1-\psi_2} > t\}$$

$\psi_2 := \min\{\rho_2, \phi_2\}$

Heuristics

- $S_0 = B_0 + B_1(0, S_0) + B_2(0, S_0)$

- $\rho_2 < \phi_2$

  ◇ $\Rightarrow S_0 \approx B_0 + (\rho_1 + \rho_2) S_0$

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- $\rho_2 < \phi_2$

  - $\Rightarrow S_0 \approx B_0 + (\rho_1 + \rho_2) S_0$

  - $\Rightarrow S_0 \approx B_0 / (1 - \rho_1 - \rho_2)$

Heuristics

- $S_0 = B_0 + B_1(0, S_0) + B_2(0, S_0)$

- $\rho_2 < \phi_2$

  - $\Rightarrow S_0 \approx B_0 + (\rho_1 + \rho_2)S_0$
  - $\Rightarrow S_0 \approx B_0/(1 - \rho_1 - \rho_2)$

- $\rho_2 \geq \phi_2$

  - $\Rightarrow S_0 \approx B_0 + (\rho_1 + \phi_2)S_0$
  - $\Rightarrow S_0 \approx B_0/(1 - \rho_1 - \phi_2)$

- **Alternative scenarios ...**

Heuristics

$B_1(0, S_0) \approx \rho_1 S_0$ **and** $B_2(0, S_0) \approx \min\{\rho_2, \phi_2\} S_0$ **because**

- $B_i(0,t) = W_i(0) + A_i(0,t) - W_i(t)$
- **Large deviations**
  - $A_i(0, S_0) \approx \rho_i S_0$

Heuristics

$B_1(0,S_0) \approx \rho_1 S_0$ **and** $B_2(0,S_0) \approx \min\{\rho_2,\phi_2\} S_0$ **because**

- $B_i(0,t) = W_i(0) + A_i(0,t) - W_i(t)$

- **Large deviations**

  ◇ $A_i(0,S_0) \approx \rho_i S_0$

  ◇ $W_i(0) = o(B_0)$

Heuristics

$B_1(0, S_0) \approx \rho_1 S_0$ **and** $B_2(0, S_0) \approx \min\{\rho_2, \phi_2\} S_0$ **because**

- $B_i(0,t) = W_i(0) + A_i(0,t) - W_i(t)$
- **Large deviations**
  - ◇ $A_i(0, S_0) \approx \rho_i S_0$
  - ◇ $W_i(0) = o(B_0)$
  - ◇ $W_i(t) \leq W_i^{\phi_i}(t)$**, also for class 1!**

Heuristics

$B_1(0, S_0) \approx \rho_1 S_0$ **and** $B_2(0, S_0) \approx \min\{\rho_2, \phi_2\} S_0$ **because**

- $B_i(0,t) = W_i(0) + A_i(0,t) - W_i(t)$

- **Large deviations**

  ◇ $A_i(0, S_0) \approx \rho_i S_0$

  ◇ $W_i(0) = o(B_0)$

  ◇ $W_i(t) \leq W_i^{\phi_i}(t)$, **also for class 1!**

  ⋆ $\rho_i < \phi_i \Rightarrow W_i(S_0) = o(B_0)$

Heuristics

$B_1(0,S_0) \approx \rho_1 S_0$ **and** $B_2(0,S_0) \approx \min\{\rho_2,\phi_2\}\, S_0$ **because**

- $B_i(0,t) = W_i(0) + A_i(0,t) - W_i(t)$

- **Large deviations**

  ◇ $A_i(0,S_0) \approx \rho_i S_0$

  ◇ $W_i(0) = o(B_0)$

  ◇ $W_i(t) \le W_i^{\phi_i}(t)$, **also for class 1!**

  ⋆ $\rho_i < \phi_i \Rightarrow W_i(S_0) = o(B_0)$

  ⋆ $\rho_2 > \phi_2 \Rightarrow B_2(0,S_0) \approx \phi_2 S_0 \Rightarrow W_2(S_0) \approx (\rho_2 - \phi_2) S_0$

Discussion

- $\rho_1 < \phi_1$
  - ◇ **Only condition on class 2: existence of** ρ_2
  - ◇ **Reduced-load equivalence**

- $\rho_1 > \phi_1$
  - ◇ **"Induced burstiness"**
  - ◇ **Unless** $\mathrm{P}\{\sup_{t\geq 0}\{A_2(0,t) - \phi_2 t\} > x\} = o(\bar{B}_1(x))$ **as** $x \to \infty$
  - ◇ **Strict priorities**

Proof using lower & upper bounds

- **"Permanent" class-1 customer arriving at time 0**

- $B_0(0,t)$ **= amount of service received by one customer**

$$\mathbf{P}\{\mathbf{S}_0 > t\} = \mathbf{P}\{\mathbf{B}_0 > B_0(0,t)\}$$

- **Identity** $B_0(0,t) + B_1(0,t) + B_2(0,t) = t$

- $W(t) := W_1(t) + W_2(t)$ **= total backlog in the system not including permanent customer**

Reference system

- **M/G/1 PS with capacity ϕ_1**
  - ◇ **Arrivals**
  - ◇ **Service requirements**
  - ◇ **Permanent customer at time 0**

Lemma At any time t each customer present in the GPS system has received at most the same amount of service in the reference system

- $W_1(t) \leq W_1^{\phi_1}(t)$
- $B_0(0, t) \geq B_0^{\phi_1}(0, t)$
- $\mathbf{S}_0 \leq \mathbf{S}_0^{\phi_1}$

First bound

- **M/G/1 PS queue with permanent customer(s) is stable**

$$\mathbf{P}\{\mathbf{W}_1^{\phi_1} \leq x\} := \lim_{t\to\infty} \mathbf{P}\{W_1^{\phi_1}(t) \leq x\}$$

- $c < \rho_2$: $Z_2^c(s) := \sup_{u \geq s} \{c(u-s) - A_2(s,u)\}$ **has a proper non-defective distribution**

Lemma

$$B_0(0,t) \leq (1-\rho_1-\psi_2+2\epsilon)t + (\rho_1-\epsilon)t - A_1(0,t) + W_1^{\phi_1}(t) + Z_2^{\psi_2-\epsilon}(0)$$

- **Sample path upper bound for** $B_0(0,t)$
- **Lower bound for** S_0
- $1-\rho_1-\psi_2 =$ **average service rate for permanent customer**

Proof

$$\begin{aligned} B_0(0,t) &\le t - A_1(0,t) + W_1(t) - B_2(0,t) \\ &\le t - (\rho_1 - \epsilon)t + (\rho_1 - \epsilon)t - A_1(0,t) + W_1^{\phi_1}(t) - B_2(0,t) \end{aligned}$$

$s \in [0,t]$: $W_2(s) = 0$ **and** $W_2(u) > 0$ **for** $u \in (s,t)$

$$\begin{aligned} B_2(0,t) &\ge A_2(0,s) + \phi_2(t-s) \ge A_2(0,s) + \psi_2(t-s) \\ &\ge (\psi_2 - \epsilon)t + A_2(0,s) - (\psi_2 - \epsilon)s \ge (\psi_2 - \epsilon)t - Z_2^{\psi_2 - \epsilon}(0) \end{aligned}$$

□

Lower bound sojourn time

Theorem: If $B_1(\cdot) \in \mathcal{I}R$ and $\rho_1 < \phi_1$, then

$$\liminf_{t\to\infty} \frac{\mathbf{P}\{\mathbf{S}_0 > t\}}{\mathbf{P}\{\mathbf{B}_0 > (1-\rho_1-\psi_2)t\}} \geq 1$$

Proof

$$\mathbf{P}\{\mathbf{S}_0 > t\} = \mathbf{P}\{\mathbf{B}_0 > B_0(0,t)\}$$

$$\geq \mathbf{P}\{\mathbf{B}_0 > (1-\rho_1-\psi_2+2\epsilon)t + (\rho_1-\epsilon)t - A_1(0,t) + W_1^{\phi_1}(t) + Z_2^{\psi_2-\epsilon}($$

$$\geq \mathbf{P}\{\mathbf{B}_0 > (1-\rho_1-\psi_2+4\epsilon)t\}$$
$$\times \underbrace{\mathbf{P}\{(\rho_1-\epsilon)t - A_1(0,t) + W_1^{\phi_1}(t) \leq \epsilon t\}}_{\longrightarrow 1} \underbrace{\mathbf{P}\{Z_2^{\psi_2-\epsilon}(0) \leq \epsilon t\}}_{\longrightarrow 1}$$

- $A_1(0,t)$ **and** $W_1^{\phi_1}(t)$ **not independent**

$$\begin{aligned} & \mathbf{P}\{(\rho_1-\epsilon)t - A_1(0,t) + W_1^{\phi_1}(t) \leq \epsilon t\} \\ \geq\ & \mathbf{P}\{A_1(0,t) \geq (\rho_1-\epsilon)t,\, W_1^{\phi_1}(t) \leq \epsilon t\} \\ \geq\ & \mathbf{P}\{A_1(0,t) \geq (\rho_1-\epsilon)t\} - \mathbf{P}\{W_1^{\phi_1}(t) > \epsilon t\} \end{aligned}$$

- $\mathbf{P}\{A_1(0,t) \geq (\rho_1-\epsilon)t\} \to 1$

- $\mathbf{P}\{W_1^{\phi_1}(t) > \epsilon t\} \to 0$

Finally, use $B_1(\cdot) \in \mathcal{IR}$

Second bound

- **Sample path lower bound for** $B_0(0, t)$
- **Upper bound for** S_0
- **Needed if** $\rho_2 < \phi_2$

Lemma

$$B_0(0, t) \geq t - W(0) - A(0, t)$$

Proof

$$B_0(0, t) = t - B_1(0, t) - B_2(0, t) = t - W(0) - A(0, t) + W(t)$$

□

Upper bound sojourn time

Theorem: If $B_1(\cdot) \in \mathcal{IR}$ **and** $\rho_1 < \phi_1$ **then**

$$\limsup_{t\to\infty} \frac{\mathbf{P}\{\mathbf{S}_0 > t\}}{\mathbf{P}\{\mathbf{B}_0 > (1-\rho_1-\psi_2)t\}} \leq 1$$

$\psi_2 = \min\{\rho_2, \phi_2\}$

Proof

- $\rho_2 \geq \phi_2 = \psi_2$:

$$\mathbf{S}_0 \leq \mathbf{S}_0^{\phi_1} \approx \frac{1}{\phi_1 - \rho_1} B_0 = \frac{1}{1-\phi_2-\rho_1} B_0$$

upper bound sojourn time ...

- $\rho_2 = \psi_2 < \phi_2$:

$$\mathbf{P}\{\mathbf{S}_0 > t\}$$

$$\leq \mathbf{P}\{\mathbf{S}_0^{\phi_1} > t,\, \mathbf{B}_0 > (1-\rho-\epsilon)t - W(0) + (\rho+\epsilon)t - A(0,t)\}$$

$$\leq \mathbf{P}\{\mathbf{B}_0 > (1-\rho-2\epsilon)t\} + \overbrace{\mathbf{P}\{\mathbf{S}_0^{\phi_1} > t,\, \mathbf{B}_0 \leq (\phi_1-\rho_1-\epsilon)t\}}^{=o(\mathbf{P}\{\mathbf{B}_0>(\phi_1-\rho_1-\epsilon)t\})}$$

$$+\mathbf{P}\{\mathbf{B}_0 > (\phi_1-\rho_1-\epsilon)t\}\, \underbrace{\mathbf{P}\{W(0)+A(0,t)-(\rho+\epsilon)t > \epsilon t\}}_{\longrightarrow 0}$$

$$\leq \mathbf{P}\{\mathbf{B}_0 > (1-\rho-2\epsilon)t\} + o(\mathbf{P}\{\mathbf{B}_0 > (\phi_1-\rho_1-\epsilon)t\})$$

$$= \mathbf{P}\{\mathbf{B}_0 > (1-\rho-2\epsilon)t\} + o(\mathbf{P}\{\mathbf{B}_0 > (1-\rho)t\})$$

Summary

- **Part III.A**
  - ◇ **proof of tail equivalence based on conditional moments**
  - ◇ **processor sharing (with service interruptions)**
  - ◇ **foreground-background processor sharing; shortest remaining processing time**

- **Part III.B**
  - ◇ **Delay of elastic traffic in a two-class GPS system**
  - ◇ $\rho_1 < \phi_1$: **reduced-load equivalence**
  - ◇ **Additional assumptions needed when** $\rho_1 \geq \phi_1$
  - ◇ $\phi_1 = 0$: **M/G/1 PS with strict priorities**

Heavy Tails: Performance Models and Scheduling Disciplines

Part III – Delay Asymptotics

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