FAIRNESS IN NETWORK OPTIMAL FLOW CONTROL

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ABSTRACT

In this paper we consider the problem of optimal flow control in a multiclass telecommunications environment where each user (or class) desires to optimize its performance while being 'fair' to the other users (classes). The Nash Arbitration Scheme from game theory is shown to be a suitable candidate for a fair, optimal operation point in the sense that it satisfies certain axioms of fairness and is Pareto optimal. This strategy can be realized by defining the product of individual user performance objectives as the network optimization criterion. This provides the rationale for considering the product of user powers as has been suggested in the literature of computer communication networks. It is shown that these points are unique in the throughput space and we also obtain some convexity properties for power and delays with respect to throughputs in a Jackson network.

INTRODUCTION

Flow control has traditionally been used in the context of congestion avoidance in networks. A multiclass environment arises due to multiple users in the network or due to multiple types of traffic. Different classes often have conflicting performance objectives, leading to a natural game theoretic framework for the analysis of the flow control problem. The use of game theoretic concepts in network optimization has been considered in [5], [12], [16], [18] and [19] where the emphasis was on the characterization of operating points, i.e. the throughputs of the individual classes, based on game theoretic equilibria.

In earlier work ([6], [7]) we argued that the network should be operated at Pareto optimal points since mathematically they correspond to equilibria from which any deviation will lead to the degradation in performance of at least one user or class. In this paper we use the Nash arbitration scheme, a game theory concept, to select a unique operating point among the multiple Pareto optimal points. This allows us to deal with the issue of optimality as well as fairness at the same time.

The organization of the paper is as follows: In Section 1 we discuss the issue of fairness and performance objectives and present the main result. In Section 2 we show that user performance based on the power function satisfy the requirements of Section 1. In doing so we show some new convexity results of the inverse of user power and delays in a Jackson-type network. In Section 3 we offer some concluding remarks.

1.0 PERFORMANCE MEASURES: FAIRNESS AND OPTIMALITY

In the multiclass environment the users or classes can be differentiated based on the grade of service required. Thus it makes little sense to maximize an overall network performance measure without regard to the actual performance of each user or class.

Game theory provides a natural framework for the analysis of the problem. This is not just an artifact, the advantage is that now we have a precise mathematical framework. This allows us to address the important issue of fairness as well as the proper operating points for the network. [1] and [17] are good references on game theory.

In a game theoretic setting there are two inherently different types of situations : cooperative and non-cooperative games. The non-cooperative game framework is one in which every class or user acts individually to optimize its performance measure without regard to the performance of other classes. Such a procedure leads to a Nash equilibrium point in the network [12]. This situation is important when the users act based only on local information [2], [8]. However, if the users are able to cooperate then the performance of each class or user may be made better than the performance achieved by the Nash equilibrium. This is because the cooperative equilibrium point turns out to be a Pareto optimal point and it has been shown that under general assumptions Nash equilibria are dominated by Pareto optimal points [9]. Hence it is desirable to operate the network at Pareto optimal points. With the cooperative framework as the basis we can then study the important issue of fairness.

The issue of fairness has been an important component in the design of optimal flow control schemes since it has been shown that there exist situations where a given scheme might optimize network throughput while denying access to a particular (or a set of) user(s) [10]. However, fairness is difficult to quantify in the absence of a proper framework. Loosely speaking, fairness can be thought of as a situation

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in which no individual class or user is denied access to the network or overly penalized.

In [10], Gerla and Staskausas define a notion of 'optimal fairness' in which total throughput is maximized subject to the network capacity being fairly utilized. An scheme which provides for equal sharing when the demands exceed capacity is then suggested as a fair scheme. From a game-theoretic standpoint, such a point is not special. Moreover, the tradeoff between throughput and delay is not taken into account. Several other ad hoc schemes might be proposed based on the ratios of individual demands or the precise nature of the individual performance objectives.

We now introduce a notion of fairness drawn from the cooperative game framework which has a precise mathematical interpretation which subsumes the usual assumptions as to what constitutes fairness. The most important outcome is that it leads to the optimization of a unique performance measure which is characterized completely by the individual performance measures.

The key notion of a fair strategy in cooperative game theory is the notion of the **Nash arbitration strategy** [20]. In order for a strategy to be a Nash arbitration strategy it should satisfy the axioms of fairness given below. See [17] for a discussion on the Nash arbitration scheme.

In order to state the axioms we first introduce the mathematical framework:

Consider a cooperative game of N players (users). Let each individual player i have an objective function $f_i(x)$: $X \rightarrow \Re$ where X is a convex, closed and bounded set of $\Re^N$. From the point of view of communications networks X will denote the space of throughputs. Let $u^* = [u_1^* , u_2^* , ... u_N^*]$ where $u_i^* = f_i(x^*)$ for some $x^* \in X$ denote a common agreement point which all the players agree to as a starting point for the game. In general u^* can be thought of as the vector of individual user performances which the user would like to at least achieve if they enter the game. Let $[U, u^*]$ denote the game defined on X with initial agreement point u^* where U denotes the image of the set X under f(.) i.e. f(X)=U. Let $F[., u^*] : U \rightarrow U$ be an arbitration strategy. Then F is said to be a Nash arbitration strategy if it satisfies the four axioms below:

1. Let $\phi(u) = u'$ where $u_i' = a_i u_i + b_i$ for i=1,2,...,N and $a_i > 0$, b_i are arbitrary constants. Then

$$F[\phi(U), \phi(u^*)] = \phi(F[U, u^*])$$

This states that the operating point in the space of strategies is invariant with respect to linear utility transformations.

2. The arbitration scheme must satisfy :

$$(F[U, u^*])_i \geq u_i^* \; for \; i = 1, 2, ..., N$$

and furthermore there exists no u $\in X$ such that $u_i \geq (F[U, u^*])_i$ for all i=1,2,3, ,N. This is the notion of Pareto optimality of the arbitrated solution.

3. Let $[U_1, u^*]$ *and* $[U_2, u^*]$ be two games with the same initial agreement point such that

(i) $U_1 \subset U_2$

(ii) $F[U_2, u^*] \in U_1$

Then $F[U_1, u^*] = F[U_2, u^*]$

This is called the independence of irrelevant alternatives axiom . This states that the Nash arbitration scheme of a game with a larger set of strategies is the same as that of the smaller game if the arbitration point is a valid point for the smaller game. The additional strategies are superfluous.

4. Let U be symmetrical with respect to a subset $J \subseteq \{1, 2, 3, ..N\}$ of indices (i.e. let i,j $\in$ J and i¡j, then $\{u_1, u_2, ..., u_{i-1}, u_j, u_{i+1}, ..., u_{j-1}, u_i, u_{j+1}, ..., u_N\} \in U$. If $u_i^* = u_j^*$ then $(F[U, u^*])_i = (F[U, u^*])_j$. for i,j $\in J$.

This is the axiom of symmetry which says that if the set of utilities is symmetric then for any two players if the initial agreement point corresponds to equal performance then their arbitrated values are equal.

REMARK: Note that the above axioms guarantee that no user (or class) is denied access to the network if $u^* = 0$ (provided superior points exist) and the arbitrated solution is at least as good as the Nash equilibrium if u^* is taken to be the Nash equilibrium. Thus, in particular, the axioms imply that a Nash arbitration strategy for the network in which the users have the same performance objectives will correspond to equal sharing if the set of admissible throughputs is symmetric and if the initial agreement point is chosen to be one which corresponds to equal throughputs by Axiom 4.

The following theorem (due to Stefanescu and Stefanescu [21]) characterizes the Nash arbitration scheme.

Theorem 1 (Nash Arbitration Scheme)

Let $f_i : X \rightarrow \Re$ i=1,2,...,N be concave, upper bounded, functions defined on X a convex, closed and bounded set of $\Re^N$.

Let $U = \{u \in \Re^N : \exists x \in X \; s.t. \; u \leq f(x)\}$ and $X(u) = \{x : u \leq f(x)\}$ and $X_0 = X(u^*)$.

Then the Nash arbitration scheme is given by the point which maximizes the unique function:

$$V(x) = \prod_1^N (f_i(x) - u_i^*) \quad over \; X_0$$

if X_0 contains vectors x which result in the user objectives strictly superior to u^*. If the vectors in X_0 have the property that there exist x $\in X_0$ such that only k of the individual objectives are superior to the corresponding elements of u^* then the unique function is taken as the product of the individual objectives for which there exist superior solutions. The remaining (n-k) components of u^* are the user objectives at the Nash arbitration point.

REMARK: *It is important to note that the solution in* general depends on the initial agreement point. The point where V(x) is maximized is defined as the fair network optimal operating point.

The use of the power function as the ratio of the average throughput over the average delay has been used in the context of flow control for some time [14], [11], [13]. In fact, it has been noted that the product of powers is a more appropriate optimization criterion [2] since the overall network power was not found to be suitable. In the following section we shall show that the inverse of the Power function satisfies the assumptions of the theorem and thus the above result justifies the use of the product of powers as a network optimization criterion. A by-product of the above result is the notion of decentralizability. For the power criterion it can be shown that the Nash equilibrium is Pareto inefficient and thus the non-decentralizability of the power criterion based on only local optimization is immediate since local optimization leads to a Nash equilibrium point [8]. It can be shown however that for a single M/M/1 queue the product of powers can be decentralized as in [7]. In general, we can say that a criterion is decentralizable if the Nash arbitration strategy can be implemented as a distributed procedure.

Before concluding this section it is important to note that the Nash arbitration strategy is not the only 'fair' arbitration scheme possible. In fact, standard criticisms of the Nash scheme (see Luce and Raiffa [17] for a complete discussion) led to the development of other arbitration schemes due to Raiffa [17] and Thompson [3]. However it can be shown that these other schemes correspond to Nash arbitration schemes for performance objectives obtained by linear transformations of the original objectives [3] and thus we restrict ourselves to the Nash arbitration scheme.

In the next section, we define some additional performance objectives for the design of optimal, fair flow control schemes. These will be shown to satisfy the hypothesis of the theorem and thus the existence of the Nash arbitration scheme. Moreover we shall show that the optimization results in unique points in the throughput space.

2.0 OPTIMAL FAIR SOLUTIONS : EXISTENCE AND UNIQUENESS

In this section we describe and analyse the design of an optimal, fair flow control scheme in a packet switched multiclass telecommunications environment. The performance criterion is the product of user powers (PPC) where power is defined as the ratio of the average throughput over the average delay of a particular user or class. It is shown that the stationary point for the PPC is the Nash arbitration scheme and gives rise to a unique vector of user throughputs.

2.1 Product of Powers Criterion

The product of powers (PPC) as a network performance criterion has been proposed in the context of performance oriented flow control for single class packet switched networks by Bharathkumar and Jaffe [2]. This was due to the fact that the overall network power was found to be unsuitable as it was deemed as lacking fairness properties. It was also noted that there could be difficulties associated with the non-concavity of the user power function. The results reported here show that the maximization of the PPC results in the Nash arbitration scheme and moreover the Nash arbitration scheme is unique in the space of throughputs, a strong result in light of the non-concavity.

When working with the user power function Theorem 1 cannot be directly applied to show the existence of the Nash arbitration scheme due to the non-concavity of the individual user power. We show however that the inverse of user power is convex with respect to the throughputs and using this property we show the existence of a Nash arbitration scheme. We then show that this result is also unique in the space of throughputs.

Let $S = [S_1, S_2, ..., S_N]^T$ denote the average throughputs for the N players in the network. We assume that the network is modelled as a Jackson network of M/M/1 queues with loop-free routing. We also assume that there are L links with link capacities $C = [c_1, c_2, ..., c_L]^T$.

Let $\Delta = [\Delta_1, \Delta_2, ..., \Delta_N]^T$ denote the vector of corresponding user delays.

Let $P_i = \frac{S_i}{\Delta_i}$ $i = 1, 2, ..., N$ denote the power function of user i which is defined on the set of admissible throughputs

$$U = \{S \geq 0 : 0 \leq \gamma_l \leq c_l ; l = 1, 2, L\}$$

where γ_l denotes the total throughput on link l.

Lemma 2.1

For a Jackson network with loop-free routing the inverse of the power of user i , i=1,2,...,N defined by

$$P_i^{-1} = \frac{\Delta_i}{S_i} \tag{1}$$

is convex in the space of throughputs i.e. P_i^{-1} is a convex function of S.

Proof :

To prove the assertion we first decompose the expression for inverse power in terms of the link contributions. This is because the average user delay is additive over links. Thus :

$$P_i^{-1} = \sum_l P_{il}^{-1} = \sum_l \frac{\Delta_{il}}{S_i}$$

where Δ_{il} is the contribution to the delay of user i by link l.

We now establish the convexity of P_{il}^{-1} by showing that the Hessian matrix :

$$H_{il} = \left[\frac{\partial^2 P_{il}^{-1}}{\partial S_j \partial S_k}\right]$$

is positive semi-definite.

We first consider the case of fixed routing in the network. Then the following expressions are easily shown:

$$\frac{\partial^2 \Delta_{il}}{\partial S_k \partial S_j} = \frac{2\alpha}{(c_l - \gamma_l)^3}\delta_{lj}\delta_{lk} \tag{2}$$

$$\frac{\partial \Delta_{il}}{\partial S_k} = \frac{\alpha}{(c_l - \gamma_l)^2}\delta_{lk} \tag{3}$$

$$\Delta_{il} = \frac{\alpha}{(c_l - \gamma_l)} \tag{4}$$

where α is the mean packet length and δ_{ij} denotes the Kronecker delta function. Hence by direct calculation we obtain:

$$\frac{\partial^2 P_{il}^{-1}}{\partial S_k \partial S_j} =$$

$$= \frac{1}{S_i}\frac{\partial^2 \Delta_{il}}{\partial S_k \partial S_j} - \frac{1}{S_i^2}\frac{\partial \Delta_{il}}{\partial S_j}\delta_{ik} - \frac{1}{S_i^2}\frac{\partial \Delta_{il}}{\partial S_k}\delta_{ij} + \frac{2}{S_i^3}\Delta_{il}\delta_{ij}\delta_{ik} \tag{5}$$

$$\frac{\partial^2 P_{il}^{-1}}{\partial S_k \partial S_j} =$$

$$= \frac{1}{S_i}\frac{\partial^2 \Delta_{il}}{\partial S_k \partial S_j} - \frac{1}{S_i^2}\frac{\partial \Delta_{il}}{\partial S_j}\delta_{ik} - \frac{1}{S_i^2}\frac{\partial \Delta_{il}}{\partial S_k}\delta_{ij} + \frac{2}{S_i^3}\Delta_{il}\delta_{ij}\delta_{ik} \quad (5)$$

For the purpose of notational simplicity we suppress the user index. This is legitimate since interchanging rows and corresponding columns of a matrix does not alter its character (i.e. positive semidefiniteness etc.). Thus the (1,1) element of H_{il} can be considered as $\frac{\partial^2 P_{il}^{-1}}{\partial^2 S_i}$ for any i=1,2,...,N.

Let M_1, M_2, M_k denote the leading principal minors of dimensions 1x1,2x2, ...kxk. In general $k \leq N$ since all the different classes need not share the given link. Then it is straightforward to show that

$$detM_1 = \frac{2\alpha}{S^3(c_l-\gamma_l)^3}[S^2 - S(c_l-\gamma_l) + (c_l-\gamma_l)^2] \geq$$

$$\geq \frac{2\alpha}{S^3(c_l-\gamma_l)^3}[S-(c_l-\gamma_l)]^2 \geq 0$$

for all feasible throughputs i.e. $S \geq 0$, $c_l - \gamma_l \geq 0$.

The 2nd. principal minor can be shown to be:

$$M_2 =$$

$$\frac{2\alpha}{S^3(c_l-\gamma_l)^3}\begin{bmatrix} S^2 - S(c_l-\gamma_l) + (c_l-\gamma_l)^2 & S^2 - \frac{S}{2}(c_l-\gamma_l) \\ S^2 - \frac{S}{2}(c_l-\gamma_l) & S^2 \end{bmatrix}$$

and hence

$$detM_2 = \frac{3}{2S(c_l-\gamma_l)} > 0$$

It can be shown that the higher order leading principal minors M_k for k=3,4,...,N have the property that $detM_k = 0$. This follows from the fact that the Hessian matrix has the form :

$$\begin{bmatrix} a-2b+c & a-b & a-b & .... & a-b & 0 & ... & 0 \\ a-b & a & a & .... & a & 0 & .... & 0 \\ .... & ..... & .... & ...... & .... & ... & .. & \\ a-b & a & a & a & a & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ ..... & .. & ... & .... & . & . & . & . \end{bmatrix}$$

and hence for $k \geq 3$ the minors contain repeated rows and thus are singular. The 0's in the matrix arise if class j does not use link l which user i takes. From above it also follows that all the remaining principal minors have determinant 0.

Hence it follows that H_{il} is positive semi-definite which demonstrates the convexity of P_{il}^{-1}. Since P_i^{-1} is the sum over all links l used by i it too is convex.

The convexity of P_i^{-1} in the case with random loop-free routing follows immediately since the corresponding Hessian is a convex combination of the Hessian with fixed routing.

Remark: The loop-free assumption is reasonable in communication networks where the routing is usually feedforward.

We now use the above result to show the existence of a *Nash arbitration scheme for the case of PPC.*

Theorem 2.2

Consider a Jackson network with loop-free routing with N users. Let the performance objective of each user be the power function defined by :

$$P_i(S) = \frac{S_i}{\Delta_i(S)} \quad ; i = 1,2,...,N$$

where S_i is the average throughput of user i, $\Delta_i(S)$ represents the corresponding average delay and S the vector of user throughputs.

The flow control scheme which maximizes the product of the user powers (PPC) is an optimal, fair flow control scheme in the sense that it corresponds to a Nash arbitration scheme for $-P_i^{-1}$ and given by :

$$S^* = argmax \prod_{i=1}^{N} P_i(S) \quad (6)$$

Moreover, S^* is unique, Pareto optimal and results in user powers superior to the Nash equilibrium.

Proof:

Note, from the previous Lemma, $P_i^{-1}(S)$ is convex for *each i and hence $-P_i^{-1}(S)$ is concave.* Working in the inverse power space precludes us from choosing the point $[0,0,.....,0]^T$ as the initial agreement point. Hence, we need to choose an initial agreement point u^* in the inverse power space which is achievable in the set of feasible throughputs X. Before proceeding to show that S^* indeed corresponds to *a Nash arbitration scheme we first show that S^* is Pareto* optimal and unique.

Consider the functions $P_i(S)$ and $\prod_1^N P_i(S)$. Then since $P_i(S)$ is defined on X which is compact and convex, $\prod_1^N P_i(S)$ is continuous and hence achieves its maximum on X. Since $P_i(S)$ is zero on the boundary of X it implies that *the maximum is achieved in the interior of X. Hence the* necessary condition that S^* satisfies is :

$$\nabla_S \prod_1^N P_i(S)\,|_{S^*} = 0 \quad (7)$$

Rewriting this in matrix form gives :

$$\begin{bmatrix} \frac{\partial P_1}{\partial S_1} & \frac{\partial P_2}{\partial S_1} & \cdots & \cdots & \frac{\partial P_N}{\partial S_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial P_1}{\partial S_i} & \frac{\partial P_2}{\partial S_i} & \cdots & \cdots & \frac{\partial P_N}{\partial S_i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial P_1}{\partial S_N} & \frac{\partial P_2}{\partial S_N} & \cdots & \cdots & \frac{\partial P_N}{\partial S_N} \end{bmatrix} \begin{bmatrix} \prod_{j\neq 1} P_j(S) \\ \cdots \\ \prod_{j\neq i} P_j(S) \\ \cdots \\ \prod_{j\neq N} P_j(S) \end{bmatrix} = 0$$

$$at\ S = S^* \quad (8)$$

Let J(S) denote the matrix above. Then from the observation that $P_i(S^*) \neq 0$ for all i, this implies that the matrix $J(S^*)$ is singular. But the matrix J(S) is the transpose of the Jacobian matrix for $[P_1(S), P_2(S), .., P_N(S)]^T$ and hence it implies that Det $|J(S^*)|$ is zero. But this is the necessary condition for a point to be Pareto optimal [22]. From the fact that any (see following) deviation from S^* results in at least one player with lower power it is also sufficient and thus S^* is the Pareto optimal point.

We now compare this point with the Nash equilibrium *point.* That the Nash equilibrium exists follows from the fact that the functions $P_i(S)$ are concave in their own throughputs *i.e.* *w.r.t* S_i (see Rosen [23]). The Nash equilibrium point for this case corresponds to the point $\bar{S}$ at which :

$$\frac{\partial P_i(S)}{\partial S_i}|_{\bar{S}} = 0 \qquad (9)$$

Hence, it is readily seen that the Nash equilibrium is not Pareto optimal and hence is Pareto inefficient (see [9]).

We now show the uniqueness of the point S^* in the space of admissible throughputs.

First note that from above it follows that the stationary point results in non-zero throughputs for each user. Define $\Pi(S) = \prod_{i=1}^{N} P_i(S)$. Then let S^* denote the stationary point of $\Pi(S)$ then it is easy to show that the necessary condition that S^* maximize $\Pi(S)$ is given by :

$$\frac{1}{S_i^*} = \frac{1}{\Pi(S^*)}\frac{\partial \Pi}{\partial S_i}(S^*) \quad i = 1,2,...,N \qquad (10)$$

Consider a perturbation of the point S^* given by

$$S = S^* + K\epsilon$$

for some feasible direction in the space of throughputs i.e.

$$S_i = S_i^* + k_i\epsilon \ , \ i = 1,2,...,N$$

Then the product of throughputs is given by : $\prod_1^N(S_i)$. Normalizing this at the point S^* gives the normalized product of throughputs as :

$$\prod_1^N(1+\frac{k_i}{S_i^*}\epsilon) = \prod_1^N(1+k_i\epsilon\sum_{s\cap i}\left[\frac{\sum l\cap s\cap i \frac{\alpha}{e_l^2}}{\sum l\cap s \frac{\alpha}{e_l}}\right] \qquad (11)$$

where $e_l = c_l - \gamma_l^*$ the residual flow in the link under S^* and $l \cap s \cap i$ denotes the set { users s and i which use link l}.

Similarly the product of the mean delays is given by

$$\prod_1^N\left(\sum_{l\cap i}\frac{\alpha}{(e_l - \sum_{l\cap s\cap i} k_s\epsilon)}\right)$$

Upon normalization with respect to the delays at the stationary point and rearrangement this can be written as:

$$\prod_1^N\left(1 + \frac{\sum l\cap i \sum l\cap s\cap i \frac{\alpha k_s\epsilon}{e_l(e_l - \sum_{s\cap l\cap i} k_s\epsilon)}}{\sum l\cap i \frac{\alpha}{e_l}}\right) \qquad (12)$$

Now it is easy to show that this is greater than (strict if at least one of the k_s is non-zero):

$$\prod_1^N\left(1 + \frac{\sum l\cap i \sum l\cap s\cap i \frac{\alpha k_s\epsilon}{e_l^2}}{\sum l\cap i \frac{\alpha}{e_l}}\right) \qquad (13)$$

By comparing (11) with (13) it can be easily seen that the normalized product of perturbed throughputs is less that the normalized product of perturbed delays implying that the perturbed PPC normalized around the stationary point is $<$ 1. This implies that any perturbation of the stationary point is not optimal establishing the uniqueness of the maximizing point in the throughput space.

Having established the uniqueness of the point S^* we now show that it corresponds to a Nash arbitration point for the negative inverse power.

Take as the initial agreement point u^* the point where $u_i^* = -aP_i^{-1}(S^*)$ $i = 1,2,....,N$ for $a > 1$ sufficiently large. Then u^* is a valid initial agreement point for a game played by N players with $-P_i^{-1}(S)$ as individual objectives. This is because if a is sufficiently large then $-aP_i^{-1}(S^*) < max_S - P_i^{-1}(S)$ and the maximum of the individual negative inverse powers exists since the functions are concave over a convex, compact domain of feasible throughputs X and attain the value $-\infty$ on the boundary of X. Now in order to apply Theorem 1 to the concave, upperbounded (because of the comment above) functions $-P_i^{-1}(S)$ we need to show the convexity and compactness of the set :

$$X_0(u^*) = \{S : u \in U \ s.t. \ u_i \geq u_i^* \ i = 1,2,..,N\}$$

where

$$U = \{u : \exists S \in X \ s.t. \ u_i \leq -P_i^{-1}(S) \ , \ i = 1,2,...,N\}$$

Note that by our choice of u^* , $X_0(u^*)$ is non-empty. Choose any arbitrary points u^1 *and* u^2 in U. Then to show that $X_0(u^*)$ is convex it is enough to show that U is convex. To show that U is convex we need to show that $u^3 = c\,u^1 + (1-c)\,u^2$, $for\ 0 \leq c \leq 1$ is in U.

Let S^1 *and* S^2 be two throughput vectors corresponding to u^1 and u^2 respectively. Then from the definition of U we have:

$$c(-P_i^{-1}(S^1)) + (1-c)(-P_i^{-1}(S^2)) \geq u_i^3$$

and from the concavity of $-P_i^{-1}(S)$ over X we have :
$-P_i^{-1}(S^3) = -P_i^{-1}(cS^1 + (1-c)S^2) \geq$
$\geq -cP_i^{-1}(S^1) - (1-c)P_i^{-1}(S^2) \geq u_i^3.$

The convexity of X implies that S^3 is a valid throughput and hence u^3 belongs to U. Hence, the set $X_0(u^*)$ is convex. Compactness follows from the fact that the set is closed and bounded.

Hence, applying theorem 1 to the user functions $-P_i^{-1}(S)$, $i = 1,2,...,N$ with initial agreement point u^*, the Nash arbitration scheme exists and is the point which maximizes $\prod_1^N(-P_i^{-1}(S) - u_i^*)$ over $X_0(u^*)$. The necessary conditions for this are :

$$\mathbf{J(S)}\begin{bmatrix} \frac{\prod_{j\neq 1}(-P_j^{-1}(S) - u_j^*)}{P_1(S)^2} \\ \cdots \\ \frac{\prod_{j\neq i}(-P_j^{-1}(S) - u_j^*)}{P_i(S)^2} \\ \cdots \\ \frac{\prod_{j\neq N}(-P_j^{-1}(S) - u_j^*)}{P_N(S)^2} \end{bmatrix} = 0 \qquad (14)$$

at the stationary point. $\mathbf{J}$(S) is the matrix defined above and corresponds to the transpose of the Jacobian for the vector of powers. Note that $P_i(S) \neq 0$ at the stationary point since $X_0(u^*)$ excludes the point [0,0.....,0].

Multiply the vector on the lhs of (14) by $(-1)^{N-1}(\prod_1^N P_i(S))^2$ then the lhs of (14) evaluated at the point S^* can be written as :

$$\mathbf{J(S^*)}\begin{bmatrix} [\prod_{j\neq 1} P_j(S^*)](1-a)^{N-1} \\ \cdot\cdot \\ [\prod_{j\neq i} P_j(S^*)](1-a)^{N-1} \\ \cdot\cdot \\ [\prod_{j\neq N} P_j(S^*)](1-a)^{N-1} \end{bmatrix}$$

From the definition of S^* we see that the vector $[\prod_{j\neq 1} P_j(S^*), ..., \prod_{j\neq N} P_j(S^*)]^T$ belongs to the null space of $J(S^*)$ and hence S^* satisfies the necessary condition for it to be the stationary point of $\prod_1^N(-P_i^{-1}(S) - u_i^*)$. The concavity of the functions $-P_i^{-1}(S) - u_i^*$ implies that the condition is also sufficient and hence S^* is the Nash arbitration scheme for the negative inverse powers and the proof is done.

REMARKS: The lack of concavity of the user power function presents difficulties in concluding that the point which maximizes the PPC is a Nash arbitration scheme for the power criterion. This is due to the difficulty of showing that the set U of allowable powers is convex and compact. However, several non-trivial examples have been worked out which show that the set U is in fact convex and compact. This leads us to conjecture that the maximization point of the PPC is in fact the Nash arbitration scheme for the power function with [0,0,.....,0] as the initial agreement point. For the case of M/M/1 queues the convexity and compactness of the set of achievable powers over the set of feasible throughputs has been shown in [6]. A network of two queues in tandem is analysed in the next paragraph.

Example: Consider the network of figure 1, where S_1 and S_2 are the throughputs of users 1 and 2 respectively and c_1 and c_2 are the capacities of links 1 and 2. Obviously $0 \leq S_1 < c_1$ and $0 \leq S_1 + S_2 < c_2$. In figure 2, the set of throughputs, where the products of powers is a concave function, is plotted surrounded by a shaded area that covers the space of admissible throughputs ($c_1 = 0.5$, $c_2 = 1.0$). Two particular examples of the product of powers as a function of the throughput of user 1 are shown in figures 3 and 4. In figure 3, $c_1 = 0.5$, $c_2 = 1$ and $S_2 = 0.3$, corresponding to a concave product of powers function. In figure 4, $c_1 = 0.5$, $c_2 = 1$ and $S_2 = 0.5$, corresponding to a nonconcave product of powers function. By using the fact the product of powers is unimodal, ascent algorithms were used to compute the maximal point. An iterative algorithm whose fixed point is the Nash Equilibrium was used for the calculation of the Nash Equilibrium [24]. In figure 5, these points are plotted in the space of powers. For the point $(S_1^*, S_2^*) = (0.1995, 0.3346)$ the product of powers is $P^* = P_1(S_1^*, S_2^*) \times P_2(S_1^*, S_2^*) = 5.6812 \times 10^{-3}$ and the individual powers are $P_1(S_1^*, S_2^*) = 3.644 \times 10^{-2}$ $P_2(S_1^*, S_2^*) = 1.5589 \times 10^{-1}$. For the Nash equilibrium point $(S_1^+, S_2^+) = (0.2692, 0.3564)$ the product of powers is $P^+ = P_1(S_1^+, S_2^+) \times P_2(S_1^+, S_2^+) = 5.1689 \times 10^{-3}$ and the individual powers are $P_1(S_1^+, S_2^+) = 3.8057 \times 10^{-2}$ $P_2(S_1^+, S_2^+) = 1.3582 \times 10^{-1}$. It should be noted here that the point which maximizes the product of powers corresponds to an arbitration point with initial agreement point (0,0). If we had started from the Nash equilibrium as the initial agreement point we would have reached a point with larger powers for both users than the ones at the Nash equilibrium point.

3.0 CONCLUSIONS

In this paper we have presented a precise mathematical formulation and characterization of the issue of the design criteria for network optimal flow control and the related issue of fairness. By using the game theoretic framework, we *have identified the Nash arbitration scheme as a desirable* optimal, fair operating point for the individual users. Furthermore, the strategy can be obtained by only knowing the individual user performance criteria. We have provided a proof of why the product of powers is indeed a reasonable design criterion and shown some new convexity properties of the power function and user delay functions. The concept was applied to a simple network of two queues in tandem.

These results could be thought of as the first concrete attempt at providing a mathematical basis for optimal flow control and fairness in the network context. An important issue which arises is the design of decentralized algorithms to achieve these operating points and the extension of these ideas to the general environment where there is mixed type of traffic.

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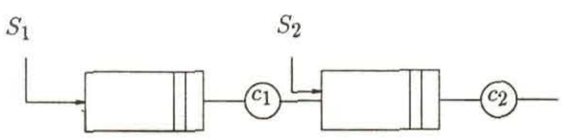

Fig. 1 A network of two queues in tandem

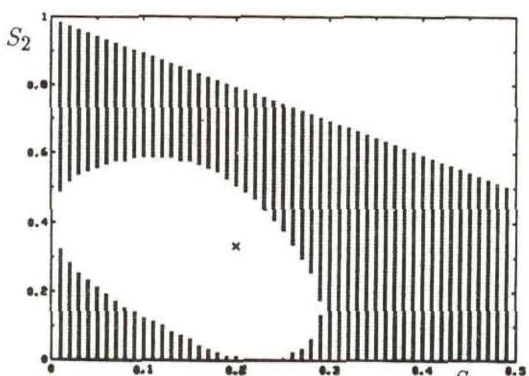

Fig. 2 Concavity region of product of powers (c_1 = 0.5, $c_2 = 1.0$)

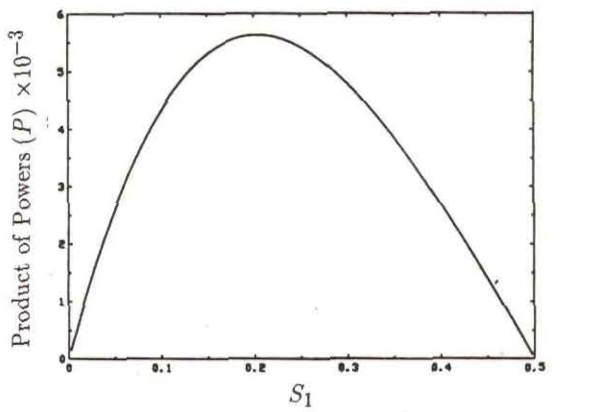

Fig. 3 Graph of product of powers for $c_1 = 0.5$, $c_2 = 1.0$, and $S_2 = 0.3$

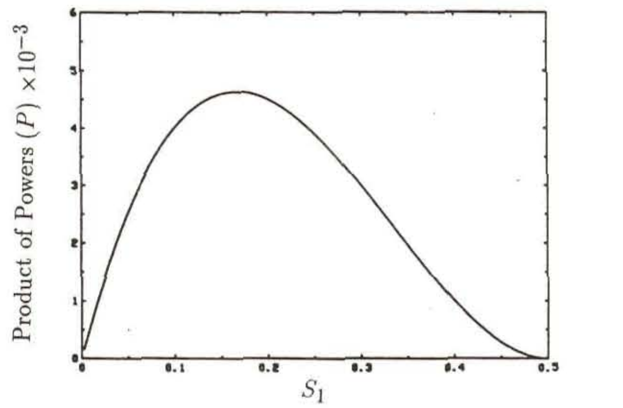

Fig. 4 Graph of product of powers for $c_1 = 0.5$, $c_2 = 1.0$, and $S_2 = 0.5$

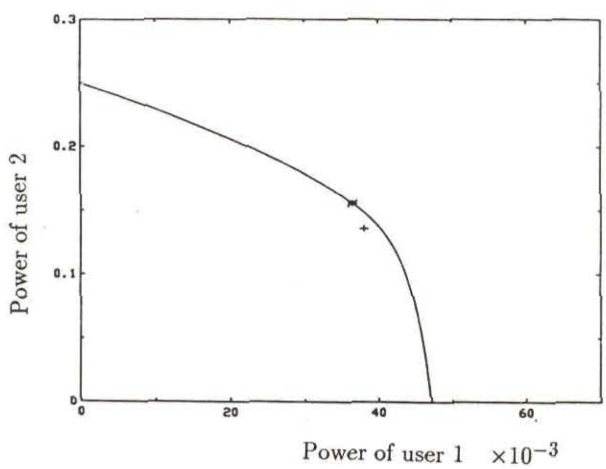

Fig. 5 Nash Equilibrium (+) and maximum of the product of powers (*) - example for $S_1 = 0.5$ and$S_2 = 1.0$